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Spectral estimates from subdivided meteorological series

by

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AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

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Abstract

The relation $\frac{1}{2}\frac{\partial^2}{\partial s^2}\left(s^2\bar{\Phi}(s)\right) = 1 - R_+(s)$

between $\bar{\Phi}(s)$, the average fraction of the covariance contained in subseries of s terms, and $R_+(s)$, the even part of the cross-correlogram, is simply derived. Its use and practical limitations are illustrated by the analysis of an artificial series.

The form $\bar{\Phi}(s) = s/(a + s)$ appears to be a suitable approximation for certain series involving vertical wind near the ground. The corresponding spectrum is discussed.

1. Introduction

Meteorologists in several countries are making estimates of turbulent fluxes and intensities by what has been called the 'direct' method. The usual technique is to compute mean products of the type

$$F(N) = (1/N)\sum_{t=1}^{N}(U_t - \bar{U})(W_t - \bar{W})$$

$$= (1/N)\sum_{t=1}^{N} u_t w_t \qquad (1)$$

Here U_t, W_t are values of appropriate meteorological variables measured at equal intervals of time over a total duration T.

$$\bar{U} = (1/N)\sum_{1}^{N} U_t \quad \text{and} \quad \bar{W} = (1/N)\sum_{1}^{N} W_t$$

F(N) is then taken to be proportional to the flux or intensity required. For instance, if U were a horizontal and W the vertical component of wind, then F(N) may be supposed to be a measure of the horizontal shearing stress in the direction of U.

Among the various theoretical and experimental difficulties which arise there is the problem of specifying the duration T over which the observations should be made. A method devised by Swinbank (see Swinbank, 1955, and Deacon, 1955) has been used to check that a chosen duration is appropriate. The record, over duration T, containing N observations, is divided into m sub-records each containing s. Mean products similar to those of Equation (1) are computed for each sub-record, and the average value of the m mean products is taken as the value of F(s):

$$F(s) = \frac{1}{N}\sum_{a=0}^{m-1}\left[\sum_{t=as+1}^{s(a+1)} u_t w_t - \frac{1}{s}\sum_{t=as+1}^{s(a+1)} u_t \sum_{t=as+1}^{s(a+1)} w_t\right] \qquad (2)$$

The process is repeated for different values of s, and $\Phi(s) \equiv F(s)/F(N)$ plotted against s/N. If $\Phi(s)$ approaches unity for $s < N$, then the duration s is deemed adequate.

In this note we are concerned to interpret this test in terms of other statistics of the variables, in particular their lag-covariance, and a suitable relation is derived by elementary methods. It is possible to derive the even part of the lag-covariance (or the co-spectrum) from the statistics of sub-records of a stationary process: an example shows that the method, though difficult in practice, can be useful.

Some of the additional difficulties associated with non-stationary meteorological series are mentioned. Further examples show that series involving the vertical velocity near the surface have special properties from which plausible and interesting results can be derived.

2. The relation between F(s) and the lag-covariogram, C(s)

We seek a relation between F(s) and C(s) where

$$C(s) = \frac{1}{N-s}\sum_{t=1}^{N-s} u_t w_{t+s} \qquad s \geq 0$$

$$= \frac{1}{N+s}\sum_{t=1}^{N+s} u_{t-s} w_t \qquad s \leq 0$$

Our derivation is by elementary argument and has no pretence to rigour.*

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* Whilst this paper was in preparation Kahn (1957) published a more general derivation.

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We write Equation (2) as

$$F(s) = F(N) - \overset{*}{F}(m) \quad , \text{ where}$$

$$Ns\,\overset{*}{F}(m) = \sum_{a=0}^{m-1} \left[\sum_{t=as+1}^{s(a+1)} u_t \sum_{t=as+1}^{s(a+1)} w_t \right] \qquad (3)$$

If we now evaluate $Ns\overset{*}{F}(m)$ by straightforward cross-multiplication we get rm terms of type $u_t\, w_{t-(s-r)}$ and $u_t\, w_{t+(s-r)}$, where $1 \ll r \ll s$.

Our best estimate for the sum of the mr samples of type $u_t\, w_{t+(s-r)}$ is rm $C(s-r)$, so that

$$\overset{*}{F}(m) \simeq \frac{1}{s}\left[\frac{1}{s}C(1-s) + \frac{2}{s}C(2-s) + \cdots + C(0) + \cdots + \frac{1}{s}C(s-1)\right] \qquad (4)$$

i.e., $F(N) - F(s)$ can be estimated from $C(s)$ by using a weighted mean.

If we consider $C(s)$ and s as continuous functions, all the summations being re-defined as corresponding integrals,

$$F(N) - F(s) \simeq \frac{1}{s}\int_{-s}^{s} C(k)\left(1 - \frac{|k|}{s}\right)dk \qquad (5)$$

We can define $C_+(s)$ as the even part of $C(s)$ by

$$2C_+(s) = C(s) + C(-s)$$

when (5) becomes

$$F(N) - F(s) \simeq \frac{2}{s}\int_{0}^{s} C_+(k)\left(1 - \frac{k}{s}\right)dk \qquad (6)$$

Formulae similar to (6) turn out to be fairly common. They are to be found in textbooks of statistics, for example that of Brooks and Carruthers (1953), for estimating the variance of mean values in a single auto-correlated series. Ogura (1950) (1952) has used a similar formula to interpret single series of atmospheric turbulent fluctuations in terms of the similarity theory of isotropic turbulence. Equation (6) can be transformed by using a spectral function $S(n) = \int_0^\infty C(s)/C(0).\cos 2\pi ns\,ds$ to

$$\Phi(s) = \int_0^\infty S(n)\,(1 - \chi^2)\,dn \qquad (7)$$

where $\chi\,(sn) = (\sin \pi sn)/\,\pi sn$.

This is the form given by Webb (1955) and attributed by him to van der Hoven and Panofsky (1954).

/Equation

Equation (6) can be written in a more useful form by multiplying by s^2 and differentiating twice with respect to s to get

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}\left(s^2 F(s)\right) = C(0) + C_+(s) \qquad (8)$$

and correspondingly

$$\frac{1}{2}\frac{\partial^2}{\partial s^2}\left(s^2 \underline{\Phi}(s)\right) = 1 - C_+(s)/C(0)$$

$$= 1 - R_+(s) \qquad (9)$$

where $R_+(s)$ is the even part of the cross-correlogram as usually defined.

3. Stationary processes: an artificial example

For the hypothetical stationary process, where statistics such as $\bar{U}$, C(s), S(n), etc. are well defined, Equation (8) shows how $C_+(s)$ can be determined from F(s).

Given a finite record which is approximately stationary, F(s) is easier to compute than C(s), but it seems likely that the double differentiation with respect to s will make the method difficult to apply in practice. We have investigated the method using an artificial series published by Kendall (1946). This series (Kendall's experimental series 1) consists of 480 terms constructed according to the formula

$$u_{t+2} = 1.1\, u_{t+1} - 0.5\, u_t + \epsilon_{t+2}$$

The element ϵ was obtained by taking two-figure random numbers between -49 and +49*.

Values of $1 - \underline{\Phi}(s)$ were computed for all integral factors of 480 and are plotted against $\log_{10}$s in Fig. 1a. The scatter is such as to preclude the direct estimation of R(s) by double differentiation of $s^2\,\underline{\Phi}(s)$. First differences were therefore read from a smooth curve drawn by eye, and approximate values of $(1/2)\,\partial/\partial(\log s)\left(1-\underline{\Phi}(s)\right)$ plotted. Another smooth curve provided estimates of $\frac{1}{2}\frac{\partial^2}{\partial(\log s)^2}\left(1-\underline{\Phi}(s)\right)$, which in turn were smoothed. Corresponding ordinates of the three smooth curves were added to give the estimated variation of R(s) shown in Fig. 1b. The values computed directly, tabulated by Kendall, are also plotted and the form of R(s) which would be expected from an indefinitely long record is also indicated. The graphical method appears to give useful smoothed results, in spite of its subjective nature. We should perhaps mention that the smooth curves of Fig. 1 were drawn by a disinterested person (the wife of one of the authors).

4. Statistics of series involving vertical wind near the surface

Meteorological series are characteristically non-stationary and statistics such as U, C(s), S(n) vary with T, the total duration of the series available for analysis, as well as in space and time. Such difficulties have

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* The mean of the series was found to be 0.821; Kendall's value of 8.821 appears to be a misprint.

been reviewed by Charnock (1957) and we shall not discuss them further here. We have analysed several sets of observations made to test the 'direct' method for estimating the flux of heat and momentum, and we have found that series which involve the vertical wind component close to (i.e., about 1 m above) the surface have, approximately, some of the attributes of stationary series. For example, $R_+(s)$ approaches zero at relatively small and practicable values of s. It may not be entirely misleading to apply methods appropriate to stationary series to the study of these particular examples.

Observations of this type are made by instruments which integrate the wind components over a certain period of time. The individual members of the series, w_t etc., are not 'instantaneous' winds, but means over a time depending on the time-constants of the equipment used. We have found that for a given N, within fairly wide limits, and for the time-constants of the apparatus we have used (over a hundred-fold range), $F(s)$ and $R_+(s)$ are approximately independent of the time-constant of the apparatus.

We can use this observed (approximate) property to establish a functional form for $F(s)$ and hence for $R_+(s)$. We have (Equation 3)

$$F(N) \equiv F(s) + \overset{*}{F}(m), \qquad N = sm$$

If we take it, from observation, that

$$\overset{*}{F}(m) \simeq F(m),$$

then $$F(N) \simeq F(m) + F(s),$$

i.e., $$1 \simeq \underset{\sim}{\Phi}(m) + \underset{\sim}{\Phi}(s) \qquad (10)$$

Equation (10) is satisfied generally by the solution $\underset{\sim}{\Phi}(s) = \log_N s$, and for a given N, in the physically significant range $2 \ll s \ll \frac{1}{2}N$, by the family $\underset{\sim}{\Phi}(s) = 1 - 1/((s^2/N)P + 1)$, which in the particular case of $p=\frac{1}{2}$ reduces to

$$\underset{\sim}{\Phi}(s) = \frac{s}{s+N^{\frac{1}{2}}} \equiv \frac{s}{s+a} \qquad (11)$$

This simple expression has proved an adequate fit to many of our observations.

Fig. 2 shows an example. The series used are of 2000 observations of vertical wind, W, and temperature, Θ at $\frac{1}{4}$-sec intervals, the individual observations being instrumental integrations over about 1/10 sec. The points plotted are of $\underset{\sim}{\Phi}(s)$ for $w\theta$ with N = 1600, the curve being $\underset{\sim}{\Phi}(s) = s/(s + 40)$. For comparison the more general solution of Equation (10), $\underset{\sim}{\Phi}(s) = \log_{1600} s$, is also plotted.

This is not an isolated example; we have speculated that Equation (11) may be of general application to similar series.

5. The frequency spectrum of series involving vertical velocity near the ground

Applying Equation (9) to Equation (11), we have

$$R_+(s) = 1 - \tfrac{1}{2}\frac{\partial^2}{\partial s^2}\left(s^2 \underset{\sim}{\Phi}(s)\right) = \frac{a^3}{(s+a)^3} = \frac{1}{(1+s')^3} \qquad (12)$$

where $s \equiv as'$ and, if Δ is the number of seconds between observations, $\Delta as' = t$. Then, writing $b = 2\pi a \Delta n$ and taking the Fourier cosine transform of Equation (12) we have

$$\frac{1}{a\Delta} S(b) = \int_0^\infty \frac{\cos bs'}{(1+s')^3} ds' = \frac{1}{2} + \frac{b^2 \cos b}{2} Ci(b) + \frac{b^2 \sin b}{2}\left(Si(b) - \frac{\pi}{2}\right) \qquad (13)$$

Equation (13) is not readily computed in the interesting range of b from the easily available tables of the sine and cosine integrals Si(b) and Ci(b), and we are indebted to the Mathematics Division of the National Physical Laboratory for the tabulation which is illustrated in Figs. 3 and 6. The expression is asymptotic to $3/b^2$. Equation (13) relates the frequency spectrum of a series (or of course the co-spectrum of two series) for which Equation (12) holds to the value of a, where a^2 is the number of terms in the series or the number of terms at which the steady value of flux or intensity is attained, if this occurs.

6. Some examples

Fig. 4a shows $\Phi(s)$ for a series of 300 observations at 1/30-sec intervals of the vertical wind integrated over 1/60 sec. The series was taken at about 1.25 m above short grass, in a mean wind about 2.5 m sec^{-1}, with a heat flux of about 10 mw cm^{-2} upwards. The series fits reasonably well to Equation (11) with a = 10 or a = 15. In Fig. 4b the curves show Equation (13) with these values for a, the points being the computed Fourier cosine transform of a curve drawn smoothly through values of the lag correlation R(s) which had been computed directly from the series.

Similar agreement can be found with a computed auto-correlogram given by Jones (1957). The three series analysed, each of 240 terms, were observations of $2\frac{1}{2}$-sec averages of wind inclination to the horizontal and were made at 2000 ft above Cardington. Fig. 5a shows values of $\bar{\Phi}(s)$ computed from the original data which were kindly provided by Mr. Jones. The curve drawn is of $s/(s + a)$ with $a = (240)^{\frac{1}{2}}$. Fig.5b shows the corresponding auto-correlogram, the directly computed values of R(s) for one of the series (No. 2), and the mean of R(s) for the three series.

Other observations which involve the vertical velocity at considerable height have been obtained by Crane and Chilton (1956) from an aircraft at 1700 ft above Chesapeake Bay. They display their results in the form of the composite spectrum shown in Fig. 6. The smooth curve is of Equation (13); it provides a reasonable interpretation of the observations.

7. Tentative extrapolation to higher frequencies

There is considerable interest in the form of the spectrum of the turbulence of the wind near the ground and several attempts have been made (Taylor, 1955, Macready, 1953, Webb, 1955) to demonstrate the existence of the form $S(k) \propto k^{-5/3}$ for the wave-number spectrum. This form follows by dimensional argument from the hypotheses of Kolmogoroff. Our observations and arguments have been in terms of frequency spectra, but we assume, with others, that they can be made applicable to wave-number spectra by multiplication by a constant velocity*.

The spectrum we suggest does approximate to $k^{-5/3}$ over an appreciable range, but appears to be asymptotic to k^{-2}. It must however be emphasised that we have computed the spectra of observed series, not necessarily the true turbulence spectrum, and that the observed properties of these series, which we have summarised in Equation (10), are such that the same form of spectrum, displaced appropriately along the frequency axis, is found whatever the time scale of observation, within the fairly wide (100-fold) limits we have used. This is another aspect of the facts summarised in Webb's (1955) remark "A value = (-) 5/3 ..... is clearly indicated for eddy scales ..... where it is certainly believed isotropy does not extend." It shows the danger of regarding the statistics of observed series as statistics of turbulence, unless there is independent evidence that the turbulence statistics are well defined.

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* Priestley and Sheppard (1952) and Robinson (1953) argued on dimensional grounds that frequency and wave-number spectra in the 'inertial sub-range' should differ. This appears to be in error. Knowledge of frequency only gives no knowledge of a typical length of the field, so viscous dissipation cannot be expressed in terms of frequency only and the dimensional argument as applied to frequency spectra is invalid.

A spectrum of type k^{-2} is of considerable physical interest, as it implies an equipartition of the viscous dissipation over the range of eddy sizes for which it holds. Our own observations suggest such a spectrum over a limited range: they certainly do not establish it.

The observations of Crane and Chilton (1956), however, extend to somewhat higher frequency and were made at greater heights. It seems reasonable to interpret their observations as showing a k^{-2} type of spectrum over a hundred-fold range of k and down to the short wavelength limit of their observations.

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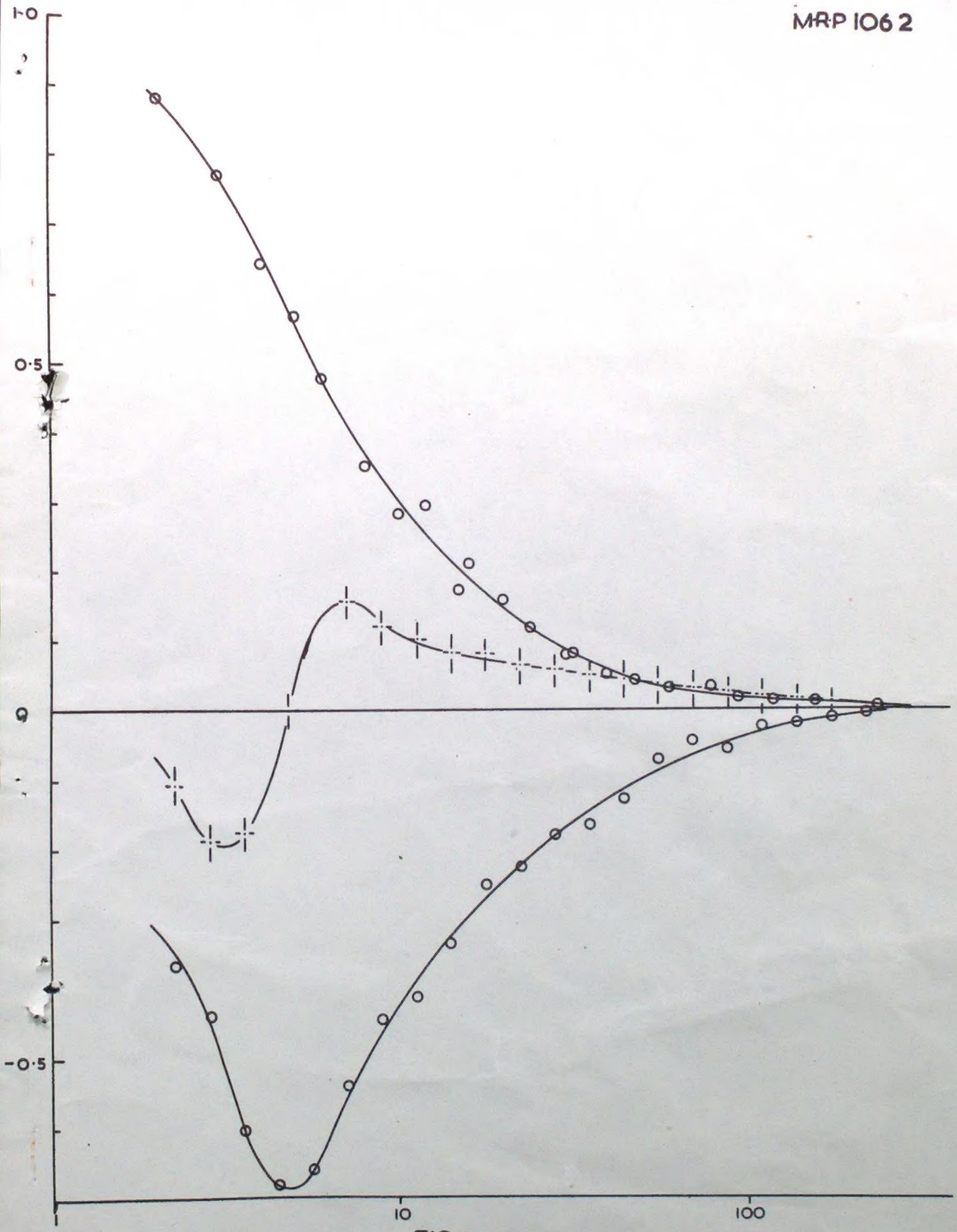

FIG. 1a.

ANALYSIS OF AN ARTIFICIAL SERIES (KENDALL No 1)

$\left\{1-\bar{\Phi}(s)\right\}$, $\frac{3}{2}\frac{\partial}{\partial(\log s)}\left\{1-\bar{\Phi}(s)\right\}$, $\frac{1}{2}\frac{\partial^2}{\partial(\log s)^2}\left\{1-\bar{\Phi}(s)\right\}$ AGAINST Log s

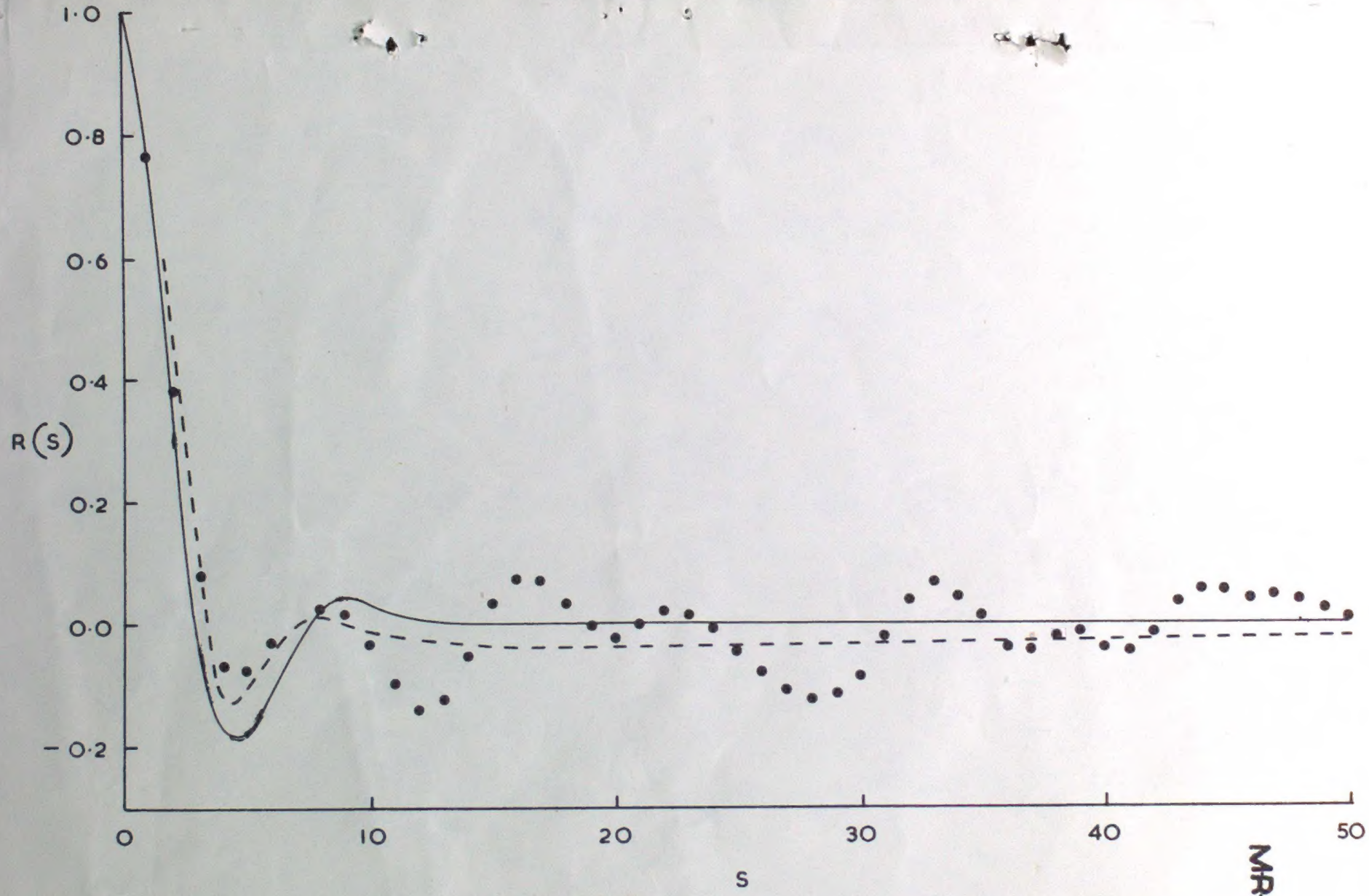

FIG. 1b.

ANALYSIS OF AN ARTIFICIAL SERIES (KENDALL No 1)

R(s) FOR INFINITE SERIES ——— BY PRESENT ANALYSIS - - - - - -

BY DIRECT COMPUTATION (KENDALL) ●●●●●

MRP 1062

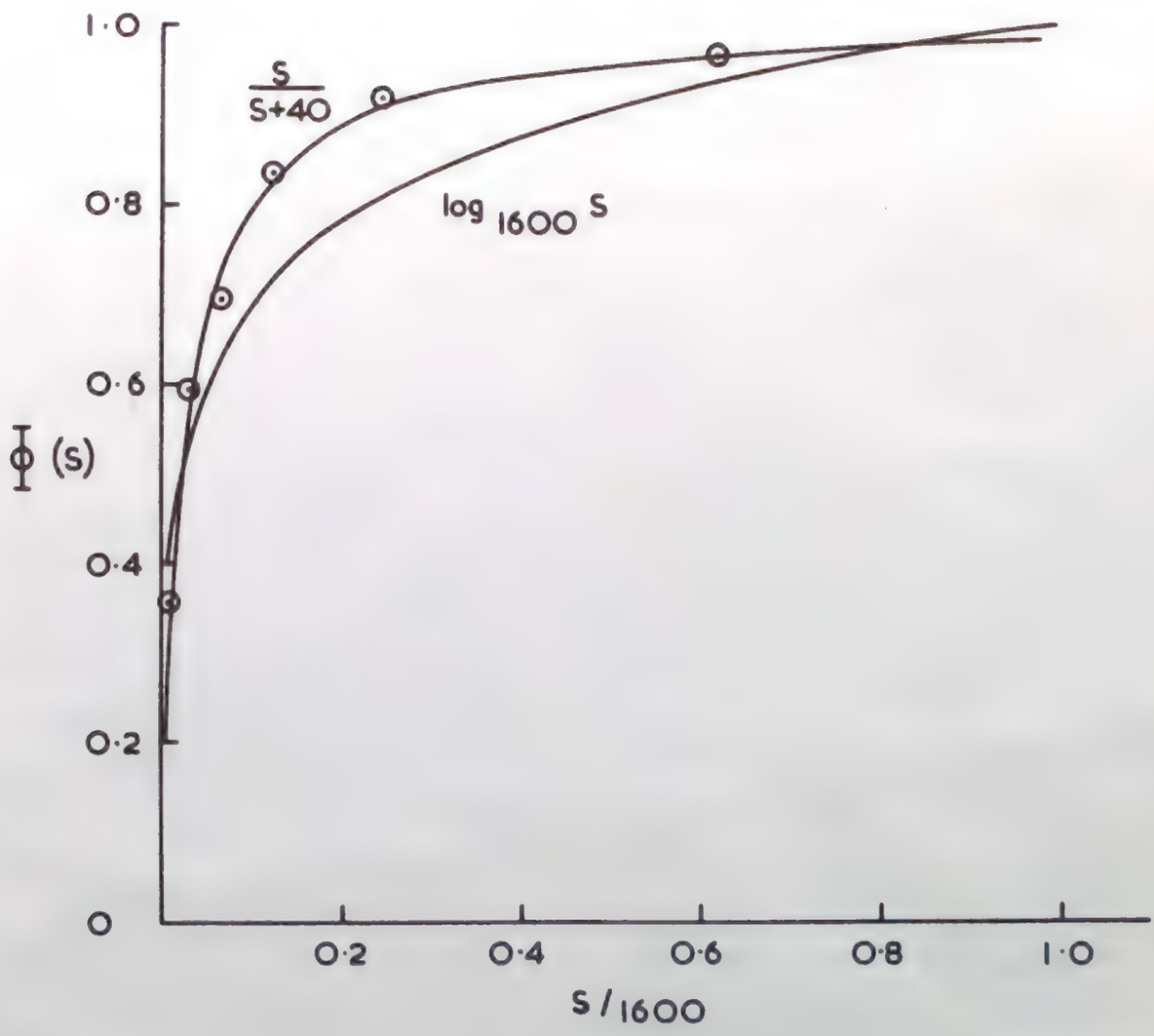

FIG.2.

Φ(s) AGAINST s/a^2 a = 40. 2000 OBSERVATIONS AT Im ΘF PRODUCT wθ. INTERVAL ¼ Sec. CARDINGTON 31 AUGUST 1954.

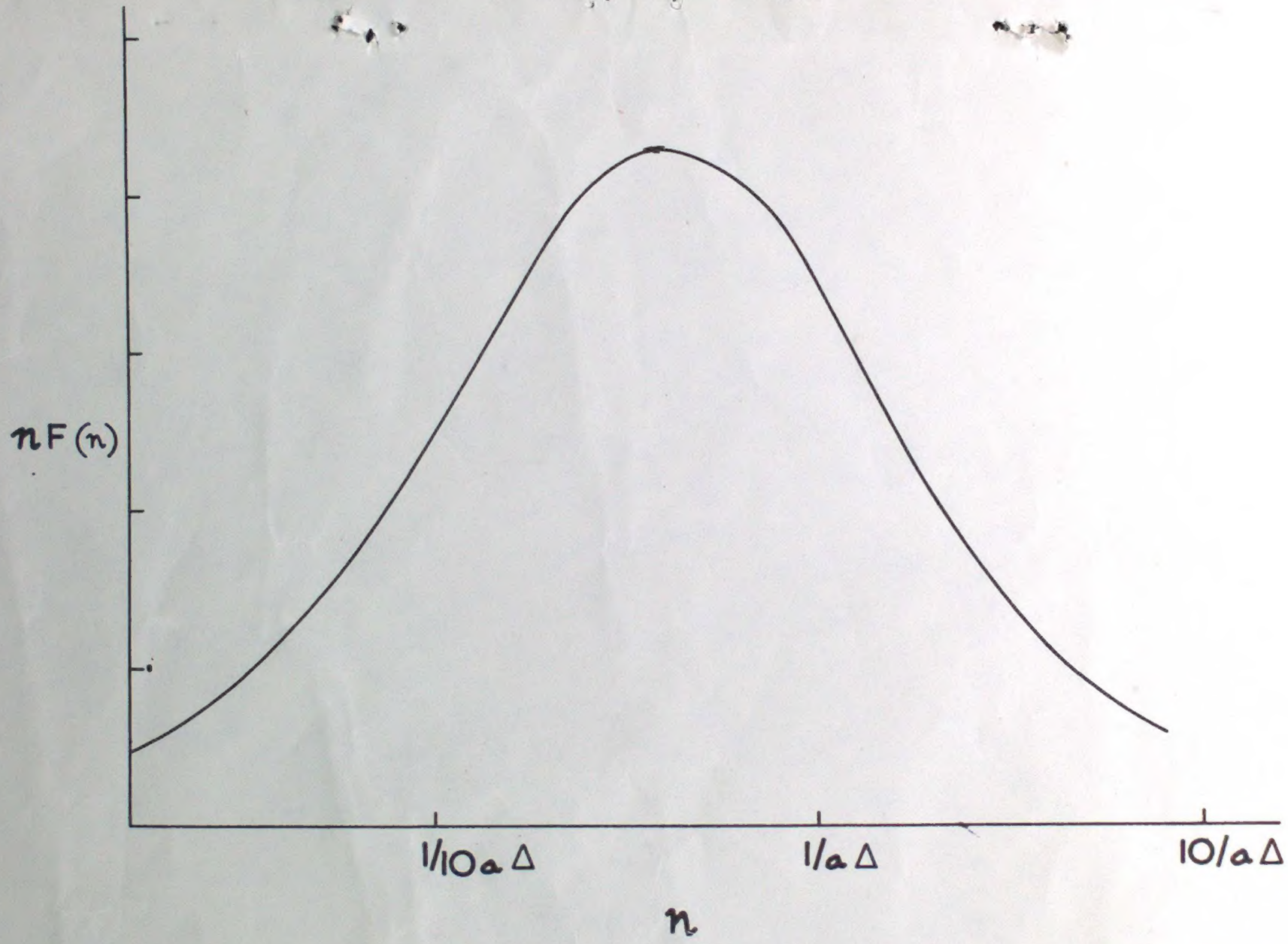

FIG. 3.

PROPOSED FREQUENCY SPECTRUM OF W ETC. (Eq. 13)

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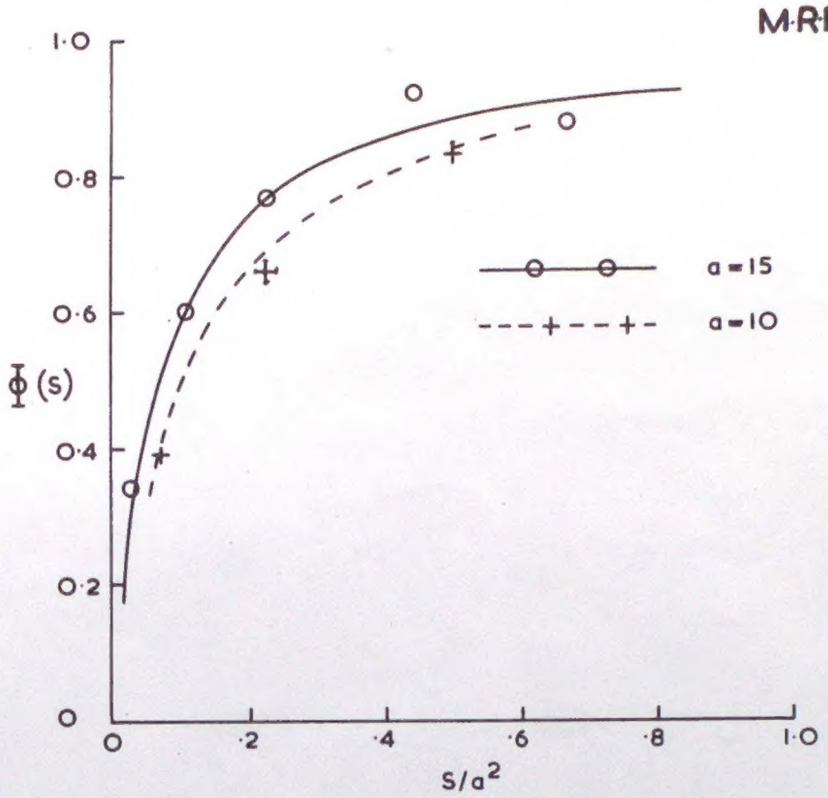

FIG. 4a.

FREQUENCY SPECTRUM OF W. 300 OBSERVATIONS AT 1m. INTERVAL 1/30 SEC. KEW 9 JUNE 1953
Φ(s) AGAINST s/a²

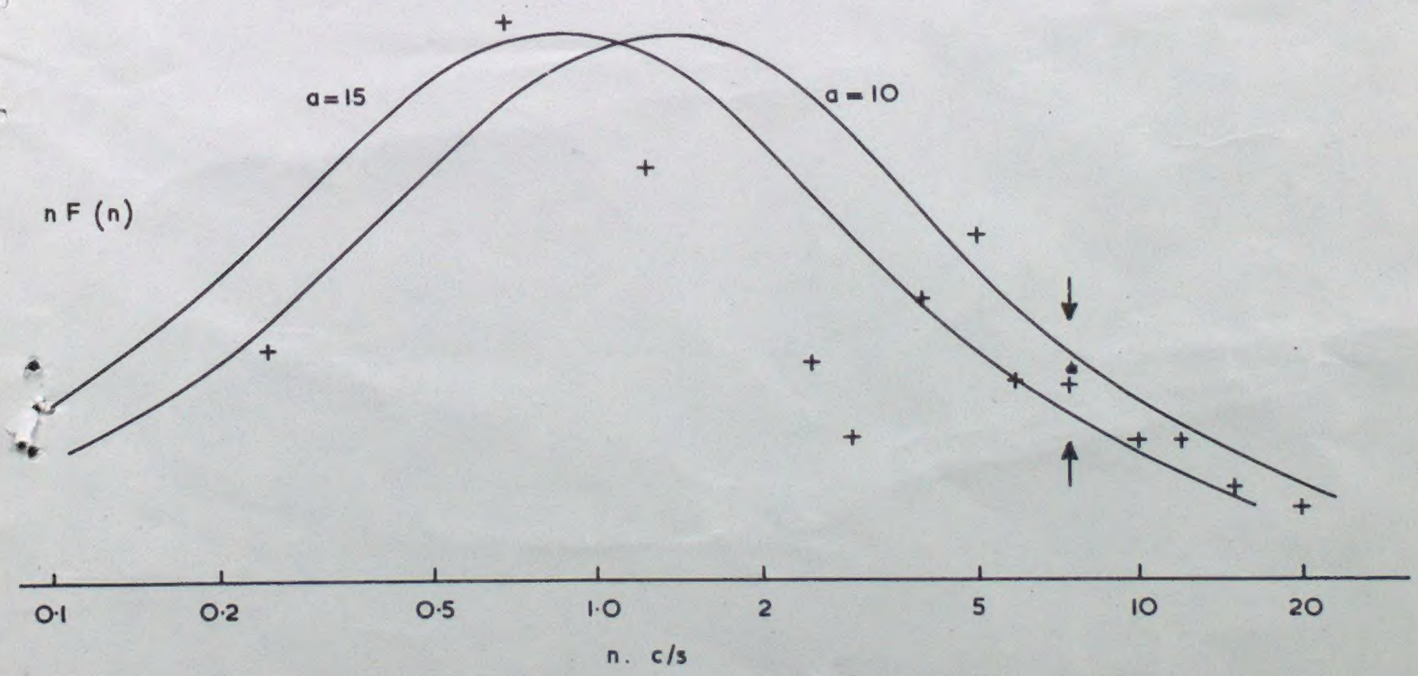

FIG. 4 b.

FREQUENCY SPECTRUM OF W. 300 OBSERVATIONS AT 1m. INTERVAL 1/30 SEC. KEW 9 JUNE 1953
FREQUENCY SPECTRUM. DIRECTLY COMPUTED POINTS FITTED IN POSITION INDICATED BY ARROW.

FIG 5. LAG CORRELOGRAM OF INCLINATION OF WIND TO HORIZONTAL.
3 SERIES OF 240 OBSERVATIONS AT 2000 ft. CARDINGTON (JONES 1957)

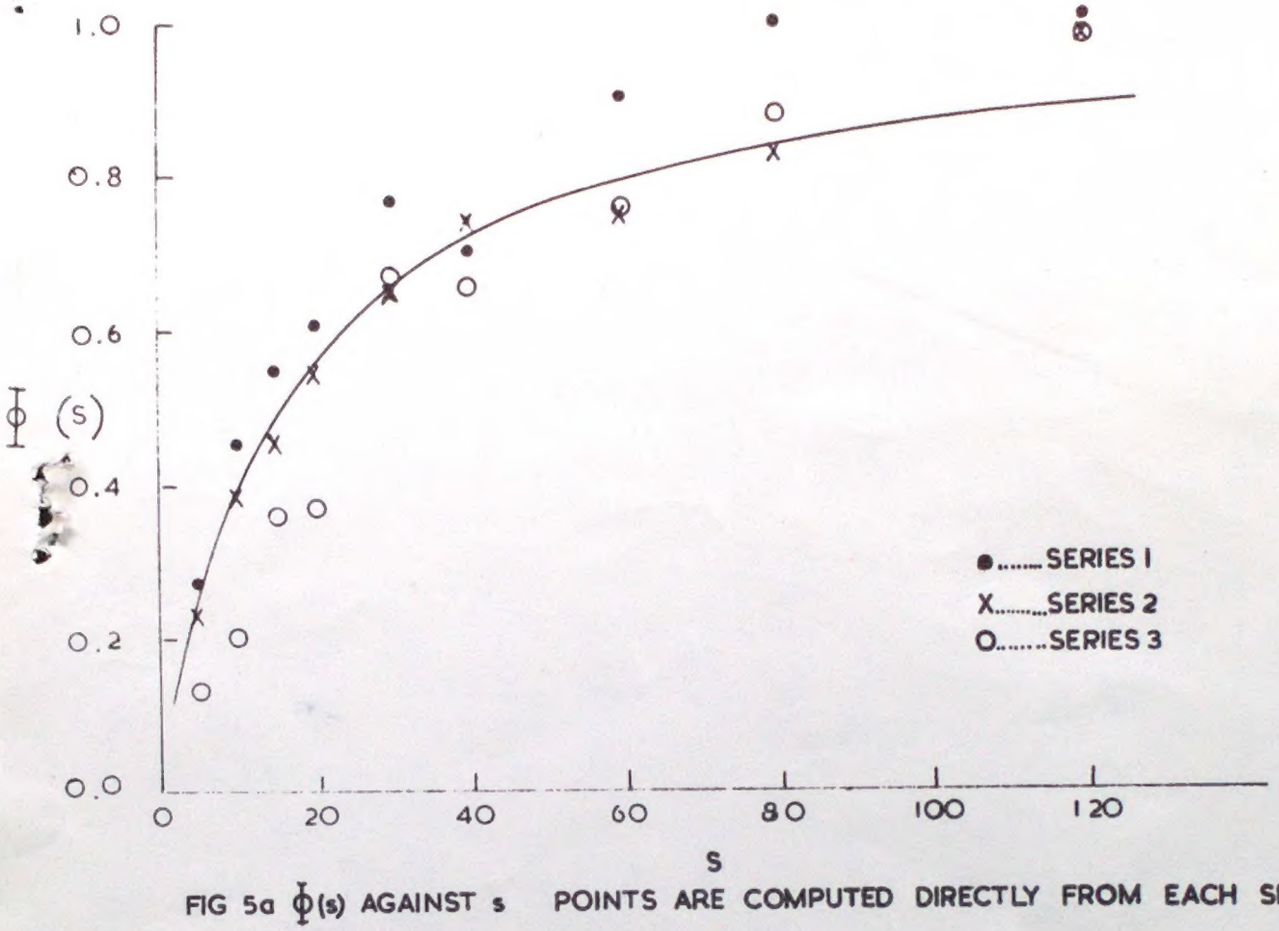

FIG 5a $\Phi(s)$ AGAINST s POINTS ARE COMPUTED DIRECTLY FROM EACH SERIES

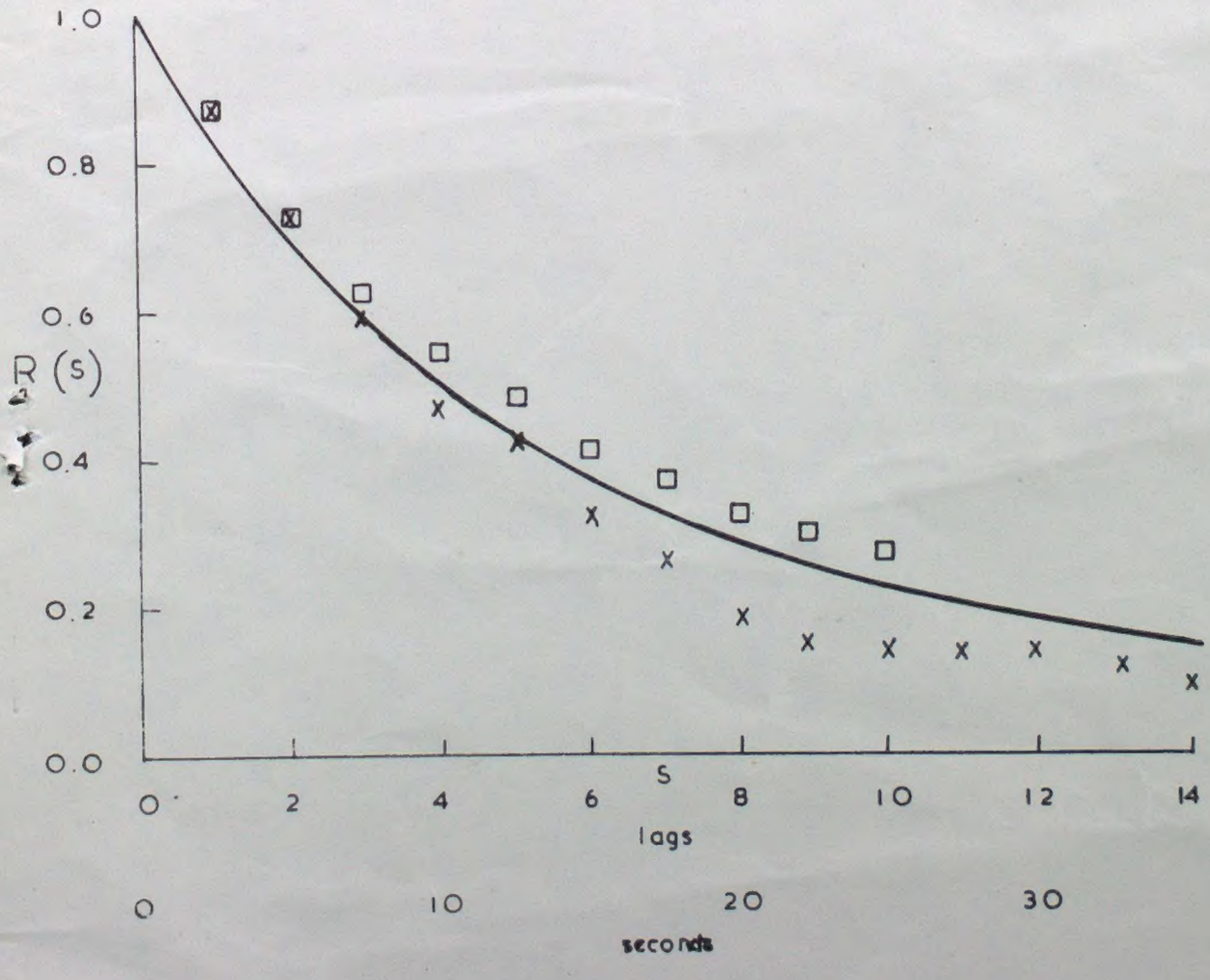

FIG 5b LAG CORRELOGRAM. CURVE BY Eq. 12 WITH $a=(240)^{\frac{1}{2}}$
POINTS ARE COMPUTED DIRECTLY FROM THE OBSERVATIONS
X..... SERIES 2. □.....MEAN OF THE THREE SERIES

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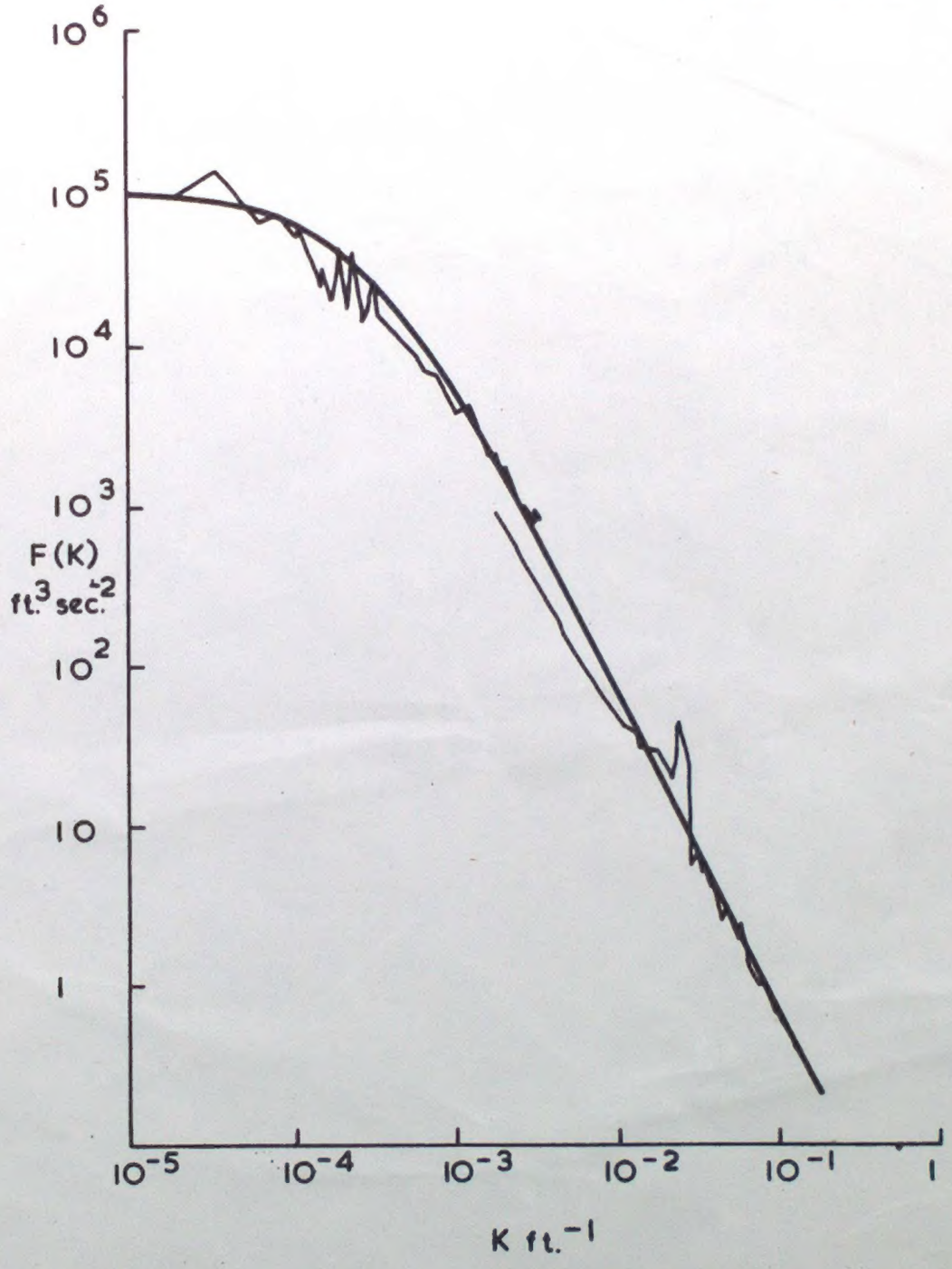

FIG 6 FREQUENCY SPECTRUM OF W AT 1700ft (CRANE AND CHILTON 1956) SMOOTH CURVE IS Eq. 13